Practical Higher-Order Unification with On-the-Fly Raising

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[Based on work with Natalie Linnell]

Motivating Higher-Order Pattern Unification

The following queries illustrate different levels of unification:

```
?- append (a::b::nil) (a :: nil) L.
   L = a :: b :: a :: nil.
```

```
?- append (a :: b :: nil) (a :: nil) (F a).
```

requires solving the unification problem

```
   (F a) = a :: b :: a :: nil
```

[multiple solutions, branching in unification]

```
?- ∀a append (a::b::nil) (a::nil) (F a).
```

requires solving

   $\exists F \forall a$ `(F a) = a::b::a::nil.`

[most general unifier, non-branching search]

The last is an instance of higher-order pattern unification.

Features of Higher-Order Pattern Unification

- Arises naturally in computations over higher-order abstract syntax
- Mixed quantifier prefixes are an essential component of the problem and usually evolve dynamically
- Has properties similar to first-order unification
  - most general unifiers can be provided
  - unification is decidable and near linear-time algorithm exists

Question: How close can we get to first-order like treatment in an implementation?

Talk Outline

- Formal presentation of the problem
- Naive, transformation rules based algorithm
- Eliminating quantifier prefixes
- Sketch of a more sophisticated algorithm based on
  - recursive traversal of terms
  - on-the-fly application of pruning and raising
- Comparison with other approaches
- Concluding comments

The Structure of Unification Problems

Universal, existential and abstracted variables are distinguished.

In particular, terms are given by

$$t ::= x \mid u \mid i \mid \lambda(i, t) \mid t(\bar{t})$$

where i is a positive number and $\bar{t}$ is a sequence of terms.

Unification problems are lists of equations under a quantifier prefix.

Examples of such problems are

$$\forall u \exists x (x = u :: nil)$$
$$\exists x \forall u (x = u :: nil)$$
$$\forall u \exists x_1 \exists x_2 (x_1 = x_2 :: nil)$$
$$\forall u_1 \forall u_2 \exists x (x(u_2) = u_1(u_2) :: nil)$$

Note: All existential, universal and lambda bound variables must be explicitly bound in the prefix or by an abstraction.

Solutions to Unification Problems

- A term t is *proper* for existential variable x if every "free" variable in it is bound outside the scope of x's quantifier.
- A unifier for a unification problem is a substitution for existential variables such that
  - each pair in it is proper, and
  - it renders the terms in each equation equal modulo the β- and η-rules

  Prefix may be extended with existential quantifiers over new variables in the process.
- A unifier is *most general* if any other unifier can be obtained from it by composition with a proper substitution.

Examples

- $\forall u \exists x (x = u :: nil)$ has $\{\langle x, u \rangle\}$ as a unifier.
- $\exists x \forall u (x = u :: nil)$ has no unifiers.
- $\forall u \exists x_1 \exists x_2 (x_1 = x_2 :: nil)$ has as a unifier $\{\langle x_1, x_3 \rangle, \langle x_2, x_3 \rangle\}$ after modification to $\forall u \exists x_3 \exists x_1 \exists x_2 (x_1 = x_2 :: nil)$.
- $\forall u_1 \forall u_2 \exists x (x(u_2) = u_1(u_2) :: nil)$ has as unifiers

  $\{\langle x, \lambda(1, u_1(1)) \rangle\}$ and $\{\langle x, \lambda(1, u_1(u_2) \rangle\}$.

  This problem has no most general unifier.

Higher-Order Pattern Unification Problems

These are problems in which the terms in the equations satisfy the following property:

Every existential variable occurrence has as arguments distinct

- lambda bound variables or
- universal variables bound within the scope of the quantifier for the existential variable.

For example, $\forall u_1 \exists x \forall u_2 (x(u_2) = u_1(u_2) :: nil)$ is such a problem.

Restriction leads to most general unifiers and decidable unification.

E.g. the problem shown as $\{\langle x, \lambda(1, u_1(1))\rangle\}$ as an mgu.

Solving Unification Problems

- Algorithm based on transformation rules of the form

  $$\langle \mathcal{Q}_1(E_1), \theta_1 \rangle \longrightarrow \langle \mathcal{Q}_2(E_2), \theta_2 \rangle$$

  such that if $\langle \mathcal{Q}(E), \emptyset \rangle \stackrel{*}{\longrightarrow} \langle \mathcal{Q}'(nil), \theta \rangle$ then θ is an mgu for $\mathcal{Q}(E)$

- Rules assume symmetry of $=$ and normal forms for terms

- Higher-order pattern restriction is assumed to be satisfied

- Transformation system is complete for higher-order pattern unification:
  - successful reduction yields mgu
  - getting "stuck" indicates non-unifiability

- Equation list yields a processing order corresponding to recursion over term structure

Notation Used in Rules

- Associated with a sequence of terms $\bar{t}$:

$\|\bar{t}\|$	length of $\bar{t}$
$\bar{t}[i]$	ith element of $\bar{t}$
$\bar{t} + \bar{s}$	concatenation of $\bar{t}$ and $\bar{s}$

- Associated with sequences of distinct lambda bound and universal variables $\overline{y}$ and $\overline{z}$:
  - if $a = \overline{z}[i]$ then $a{\downarrow}\overline{z} = |\overline{z}| + 1 - i$
  - $\overline{y}{\downarrow}\overline{z} = \overline{y}[1]{\downarrow}\overline{z}, \ldots, \overline{y}[|\overline{y}|]{\downarrow}\overline{z}$, provided all elements of $\overline{y}$ appear in $\overline{z}$.
  - $\overline{y} {\cap} \overline{z}$ is some listing of the list of elements common to $\overline{y}$ and $\overline{z}$.

Simplification Transformations

- *Removing Abstractions*

$$\langle \mathcal{Q}(\lambda(n, s) = \lambda(n, t) :: E), \theta\rangle \longrightarrow \langle \mathcal{Q}(s = t :: E), \theta\rangle$$

- *Descending Under Rigid Heads*

$$\langle \mathcal{Q}(a(s_1, \ldots, s_n) = a(t_1, \ldots, t_n) :: E), \theta\rangle \longrightarrow$$
$$\langle \mathcal{Q}(s_1 = t_1 :: \ldots :: s_n = t_n :: E), \theta\rangle$$

if a is a lambda bound or universal variable.

Note that failure occurs implicitly if heads are different

Flexible-Rigid Transformation

$$\langle \mathcal{Q}_1 \exists f \mathcal{Q}_2 (f(\overline{y}) = a(t_1, \ldots t_n) :: E), \theta \rangle \longrightarrow$$
$$\langle \mathcal{Q}_1 \exists h_1 \ldots \exists h_n \exists f \mathcal{Q}_2 (h_1(\overline{y}) = t_1 :: \ldots :: h_n(\overline{y}) = t_n :: \theta'(E)), \theta' \circ \theta \rangle$$

where $\theta' = \{\langle f, \lambda(|\overline{y}|, a'(h_1(|\overline{y}|, \ldots, 1), \ldots, h_n(|\overline{y}|, \ldots, 1)))\rangle\}$

provided

- f does not appear in $a(t_1, \ldots t_n)$, and
- a is a lambda bound or universal variable such that
  - a appears in $\overline{y}$ and $a' = a{\downarrow}\overline{y}$, or
  - a is quantified in $\mathcal{Q}_1$ and $a' = a$.

Flexible-Flexible Transformation (Same Var)

$$\langle \mathcal{Q}_1 \exists f \, \mathcal{Q}_2(f(y_1, \ldots, y_n)) = f(z_1, \ldots, z_n)) :: E), \theta\rangle$$
$$\longrightarrow \langle \mathcal{Q}_1 \exists h \exists f \, \mathcal{Q}_2(\theta'(E)), \theta' \circ \theta\rangle$$

where

- $\theta' = \{\langle f, \lambda(n, h(\overline{w}))\rangle\}$ and
- $\overline{w}$ is some listing of the set $\{m + 1 - i \mid y_i = z_i \text{ for } i \leq n\}$

Flexible-Flexible Transformation (Different Vars)

- *No Intervening Universal Quantifiers*

$$\langle \mathcal{Q}_1 \exists f \mathcal{Q}_2 \exists g \mathcal{Q}_3 (f(\overline{y}) = g(\overline{z}) :: E), \theta \rangle \longrightarrow$$
$$\langle \mathcal{Q}_1 \exists h \exists f \mathcal{Q}_2 \exists g \mathcal{Q}_3 (\theta'(E)), \theta' \circ \theta \rangle$$

for $\theta = \{\langle f, \lambda(|\overline{y}|, h(\overline{u}))\rangle, \langle g, \lambda(|\overline{z}|, h(\overline{v}))\rangle\}$

where $\overline{u} = \overline{w}{\downarrow}\overline{y}$ and $\overline{v} = \overline{w}{\downarrow}\overline{z}$ for $w = \overline{y} \cap \overline{z}$

- *Raising Transformation*

$$\langle \mathcal{Q}_1 \exists f \mathcal{Q}_2 \exists g \mathcal{Q}_3 (f(\overline{y}) = g(\overline{z}) :: E), \theta \rangle \longrightarrow$$
$$\langle \mathcal{Q}_1 \exists f \exists h \mathcal{Q}_2 \exists g \mathcal{Q}_3 (f(\overline{y}) = h(\overline{w} + \overline{z}) :: \theta'(E), \theta' \circ \theta \rangle$$

where $\overline{w}$ is a listing of the variables quantified universally in $\mathcal{Q}_2$, and $\theta' = \{\langle g, h(\overline{w})\rangle\}$.

Inefficiencies in the Naive Algorithm

- Raising Transformation
  - Maintaining and examining the quantifier prefix
  - Creating large lists of arguments
  - Introducing unnecessary arguments that have to be pruned later
- Incremental substitution generation in flexible-rigid case
  - unnecessary term construction
  - repeated occurs check
- Legitimacy check for rigid head in flex-rigid case
  - requires prefix examination
  - depends also on size of argument list for flexible term

Relevance of the Quantifier Prefix

Quantifier prefix is used for the following:

- Distinguishing existential and universal variables

  Store type tags with variables

- Checking adherance to higher-order pattern condition

  Record quantifier position

  In particular, maintain l_x, the number of changes from existential to universal quantification before the quantifier for x

- Effecting the raising transformation

  Relativize raising to the arguments of the other flexible term instead

Raising without the Quantifier Prefix

Consider the equation

$$f(\overline{y}) = g(\overline{z})$$

where f and g are existential variables such that $l_f \leq l_g$.

To solve this equation, we have to transform both sides to the form

$$h(\overline{w})$$

where

h is a new existential variable, and

$\overline{w}$ consists of two parts:

- variables u in $\overline{y}$ such that $l_u \leq l_g$
- variables shared between $\overline{y}$ and $\overline{z}$.

Substitutions for f and g must be coordinated to generate this term.

Modified Flex-Flex (Different Vars) Rule

Let $\overline{y}\Uparrow g$ denote a listing of the set

$$\{u \mid u \text{ is a universal variable in } \overline{y} \text{ such that } l_u \leq l_g\}$$

Then rules for the flexible-flexible with different heads case can be replaced by

$$\langle f(\overline{y}) = g(\overline{z}) :: E, \theta\rangle \longrightarrow \langle \theta'(E), \theta' \circ \theta\rangle$$

for $\theta' = \{\langle f, \lambda(|\overline{y}|, h(\overline{q} + \overline{v}))\rangle, \langle g, \lambda(\overline{z}, h(\overline{p} + \overline{u}))\rangle\}$

where

- h is a new existential variable such that $l_u \leq l_f$,
- $\overline{p} = \overline{y}\Uparrow g$ and $\overline{q} = \overline{p}\downarrow\overline{y}$, and
- $\overline{v} = (\overline{y}\cap\overline{z})\downarrow\overline{y}$ and $\overline{u} = (\overline{y}\cap\overline{z})\downarrow\overline{z}$

assuming that $l_f \leq l_g$.

The Full Algorithm

- Based on a recursive traversal of terms in two modes:
  - First-order like term simplification
  - Variable binding, initiated by flex-flex or flex-rigid pair
- Variable binding computation is parameterized by
  - variable to be bound,
  - vector of its arguments, and
  - term constituting the other half of the equation
- Subpart of variable binding is a “make substitution” phase that returns
  - a substitution term, and
  - possible substitutions for embedded variables

Example

Consider the unification problem

$$\exists x \forall a \forall b \forall c \exists y \forall d (b(x(a,d)) = b(a(y)) :: nil)$$

After labelling of variables and dropping of the prefix this becomes

$$(b_{c(1)}(x_{v(0)}(a_{c(1)}, d_{c(2)})) = b_{c(1)}(a_{c(1)}(y_{v(1)})) :: nil)$$

Comparison with Other Algorithms

Two existing styles of algorithms:

- Based on an explicit *a priori* raising

  e.g. [Nipkow], [Qian]

  - must maintain list of all universals encountered
  - blind raising coupled with pruning of redundant variables

- explicit substitution based approach, characterized by graftable metavariables

  e.g. [Dowek, Hardin, Kirchner, Pfenning], [Pfenning, Pientka]

  - can avoid initial raising, but
  - dynamic behaviour can be akin to blind raising

Conclusions and Future Work

- Algorithm has been implemented in C and SML and used in actual systems
- Relevance of explicit substitutions needs to be better understood:
  - seems useful for delaying reduction substitution, but
  - do graftable metavariables really offer a benefit?
- Compilation issues and impact on λProlog processing model to be examined.